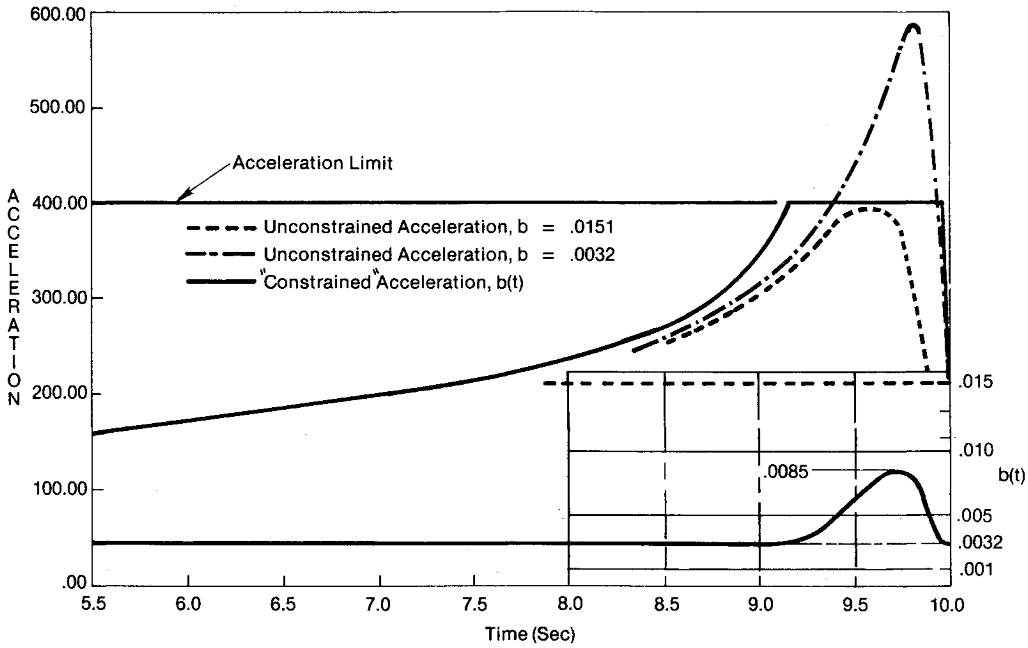


**Fig 2 Control acceleration obtained from constant and time-varying weighting functions.**

centages of energy were incurred to achieve further reductions in rms miss distance. These percentage increases in total expended energy, however, are in many cases acceptable (e.g., much more readily available than an increased constraint level for given aerodynamic limits). Further solutions were obtained at several values of the constraint levels, with qualitatively similar results.

## Conclusions

While optimal controls extremalize only a given scalar performance index, researchers have been understandably reluctant to abandon the formalism in the presence of inequality constraints which induce an additional quadratic optimality criterion with a constant weight. The performance index may then inhibit a preferred "full throttle" operation over a time interval. In applications allowing obvious physical interpretations, for example, maximum allowable control effort could be prolonged to reduce a terminal state deviation; to do so, however, would disallow using the convenient Riccati equation formalism in most linear optimization applications published to date. Replacement of constant weights by time-varying weighting functions, surprisingly rare in applications literature, is advocated here to permit usage of the full linear formalism while keeping the performance index selections more consistent with the true intent of the optimization. The rudimentary example and approach to synthesizing a time varying weighting function presented here are intended to stimulate further development for broader application.


## Acknowledgment

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# Identity Between INS Position and Velocity Error Models


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## Nomenclature

| | |
|---|---|
| $A,B,C$ | = general vectors |
| $f$ | = specific force vector |
| $g$ | = gravity vector |
| $g^m$ | = gravitation vector |
| $\Delta g$ | = error in computed $g$ vector due to error in assumed position |
| $\delta g$ | = gravity deflection and anomaly vector |
| $R$ | = position vector |
| $\Delta R$ | = position error vector |
| $V$ | = ground velocity vector |
| $\Delta V$ | = INS-computed velocity error vector |
| $\rho$ | = angular rotation rate vector of the $t$ frame with respect to the $e$ frame (see Ref. 1) |
| $\psi$ | = vector angle by which a rotation of the $c$ frame ends at the $p$ frame (see Ref. 1) |
| $\Omega$ | = Earth rate vector |
| $\omega$ | = angular rotation rate vector of the $t$ frame with respect to an inertial frame (see Ref. 1) |
| $\nabla$ | = accelerometer error vector |
| $A_q$ | = rate of change of the general vector $A$ relative to a general coordinate system $q$ |

*Subscripts*

| | |
|---|---|
| $e$ | = Earth-fixed coordinate system (see Ref. 1) |
| $i$ | = inertial coordinate system |
| $q$ | = general coordinate frame |
| $t$ | = true coordinate system (see Ref. 1) |

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## Introduction

ERROR models for inertial navigation systems (INS) have been derived in many references[1-8] and used in numerous publications to analyze pure inertial as well as aided inertial navigation systems. The differential equations which describe the error propagation in INS are divided into equations which describe the propagation of the translatory errors and equations which describe the propagation of the attitude errors.

The propagation of the translatory errors, viz. position and velocity errors, is classically described by either[1,2]

$$\Delta R_{tt}+2\omega\times\Delta R_t+\omega_t\times\Delta R+\omega\times(\omega\times\Delta R)-\Omega\times(\Omega\times\Delta R)$$
$$=\nabla-\psi\times f+\Delta g+\delta g \tag{1}$$

or[7,8]

$$\Delta V_t+(2\Omega+\omega)\times\Delta V=\nabla-\psi\times f+\Delta g+\delta g \tag{2}$$

Since Eq. (1) describes the propagation of the position error, it will here be called the "position error equation." Similarly, Eq. (2) will be called the "velocity error equation." Classically, Eq. (1) is obtained from the perturbation of the nominal position differential equation

$$R_{tt}+2\omega\times R_t+\omega_t\times R+\omega\times(\omega\times R)-\Omega\times(\Omega\times R)=f+g \tag{3}$$

which is derived from the basic accelerometer equation

$$R_{ii}=f+g^m \tag{4}$$

using the definition

$$g\triangleq g^m-\Omega\times(\Omega\times R) \tag{5}$$

The velocity error equation, i.e. Eq. (2), is obtained also from the perturbation of the following nominal velocity differential equation

$$V_t+(2\Omega+\omega)\times V=f+g \tag{6}$$

Equation (6) too is developed from Eq. (4) using Eq. (5). It is clear, then, that both the position and velocity error equations, i.e. Eqs. (1) and (2), stem from the same origin but are obtained through perturbation in different ways. Although perturbation processes involve some omissions, one would expect Eqs. (1) and (2), which relate terms containing first-order errors, to be identical. This indeed is taken for granted by the analysts who select either Eq. (1) or (2) at their convenience. It is felt, though, that the equivalence between these error equations was never proven explicitly in the open literature.

The purpose of this Note, then, is to furnish a direct proof that Eqs. (1) and (2) are identical. The proof will start with Eq. (1) and then, using known mathematical and kinematical relations, Eq. (2) will be derived.

## Derivation

Define

$$J\triangleq\nabla-\psi\times f+\Delta g+\delta g \tag{7}$$

and using this definition rewrite Eq. (1)

$$\Delta R_{tt}+2\omega\times\Delta R_t+\omega_t\times\Delta R+\omega\times(\omega\times\Delta R)-\Omega\times(\Omega\times\Delta R)=J \tag{8}$$

The gist of the proof is to eliminate $\Delta R_{tt}$, $\Delta R_t$, and $\Delta R$ from Eq. (8) and introduce $\Delta V_t$ and $\Delta V$ instead. To meet this end start with the definition

$$\Delta V\triangleq\Delta R_e \tag{9}$$

and with the following well-known relation between the time derivatives of $\Delta R$ in two frames rotating with respect to one another:

$$\Delta R_t=\Delta R_e-\rho\times\Delta R \tag{10}$$

From Eqs. (9) and (10) the following is obtained:

$$\Delta R_t=\Delta V-\rho\times\Delta R \tag{11}$$

Differentiation of Eq. (11) with respect to time in the $t$ frame yields

$$\Delta R_{tt}=\Delta V_t-\rho_t\times\Delta R-\rho\times\Delta R_t \tag{12}$$

Add and subtract $\Omega_t\times\delta R$

$$\Delta R_{tt}=\Delta V_t-(\Omega_t+\rho_t)\times\Delta R+\Omega_t\times\Delta R-\rho\times\Delta R_t \tag{13}$$

which, using

$$\omega=\Omega+\rho \tag{14}$$

becomes

$$\Delta R_{tt}=\Delta V_t-\omega_t\times\Delta R+\Omega_t\times\Delta R-\rho\times\Delta R_t \tag{15}$$

Now use Eq. (15) to eliminate $\Delta R_{tt}$ from Eq. (8):

$$\Delta V_t+\Omega_t\times\Delta R-\rho\times\Delta R_t+2\omega\times\Delta R_t+\omega\times(\omega\times\Delta R)$$
$$-\Omega\times(\Omega\times\Delta R)=J \tag{16}$$

Next, eliminate $\Delta R_t$ as follows. If $\Delta R_t$ in Eq. (16) is factored out, then using Eq. (14), Eq. (16) becomes

$$\Delta V_t+(2\Omega+\rho)\times\Delta R_t+\Omega_t\times\Delta R+\omega\times(\omega\times\Delta R)$$
$$-\Omega\times(\Omega\times\Delta R)=J \tag{17}$$

Now applying the relationship expressed in Eq. (10) it follows that

$$\Omega_t=\Omega_e-\rho\times\Omega \tag{18}$$

but obviously

$$\Omega_e=0 \tag{19}$$

therefore Eq. (18) becomes

$$\Omega_t=-\rho\times\Omega \tag{20}$$

Substitute, into Eq. (16), the expression for $\Delta R_t$ given in Eq. (11) and the expression for $\Omega_t$ given in Eq. (20). This yields

$$\Delta V_t+(2\Omega+\rho)\times(\Delta V-\rho\times\Delta R)$$
$$-(\rho\times\Omega)\times\Delta R+\omega\times(\omega\times\Delta R)-\Omega\times(\Omega\times\Delta R)=J \tag{21}$$

Now eliminate $\Delta R$ from Eq. (21). To accomplish this, start with Eq. (14) to show that

$$\omega\times(\omega\times\Delta R)-\Omega\times(\Omega\times\Delta R)$$
$$=\Omega\times(\rho\times\Delta R)+\rho\times(\Omega\times\Delta R)+\rho\times(\rho\times\Delta R) \tag{22}$$

Thus Eq. (21) becomes

$$\Delta V_t+(2\Omega+\rho)\times\Delta V-\Omega\times(\rho\times\Delta R)$$
$$-(\rho\times\Omega)\times\Delta R+\rho\times(\Omega\times\Delta R)=J \tag{23}$$

At this point, use the following vector identity:

$$A\times(B\times C)\equiv(A\cdot C)B-(A\cdot B)C \quad (24)$$

to show that

$$-\Omega\times(\rho\times\Delta R)=-(\Omega\cdot\Delta R)\rho+(\Omega\cdot\rho)\Delta R \quad (25)$$

$$-(\rho\times\Omega)\times\Delta R=(\Delta R\cdot\Omega)\rho-(\Delta R\cdot\rho)\Omega \quad (26)$$

and

$$\rho\times(\Omega\times\Delta R)=(\rho\cdot\Delta R)\Omega-(\rho\cdot\Omega)\Delta R \quad (27)$$

The sum of the right-hand side of Eqs. (25-27) vanishes; thus, summation of these equations yields

$$-\Omega\times(\rho\times\Delta R)-(\rho\times\Omega)\times\Delta R+\rho\times(\Omega\times\Delta R)=0 \quad (28)$$

Therefore, the terms containing $\Delta R$ can be eliminated from Eq. (23) which results in

$$\Delta V_t+(2\Omega+\rho)\times\Delta V=J \quad (29)$$

After the substitution of the expression for $J$ given in Eq. (7) into Eq. (29) the latter becomes

$$\Delta V_t+(2\Omega+\rho)\times\Delta V=\nabla-\psi\times f+\Delta g+\delta g \quad (30)$$

This ends the proof since Eqs. (30) and (2) are identical.

## Conclusion

The position and velocity error models given in Eqs. (1) and (2), respectively, are derived in the literature through perturbation of the respective nominal equations [Eqs. (3) and (6), respectively]. The two nominal equations stem from the same physical origin expressed in Eq. (4). Although Eqs. (1) and (2) are assumed identical, to the best of our knowledge a direct proof of this proposition was never furnished in the literature. This Note completes the triangle by providing such proof. The proof is accomplished by operations on Eq. (1) which eventually yield Eq. (2). The operations involve kinematical theorems as well as vector algebra.

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# Navigation Accuracy Analysis for an Ion Drive Rendezvous with Comet Tempel 2

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## Introduction

REFERENCES 1 and 2 have discussed various aspects of the navigation of a proposed dual comet mission using the Solar Electric Propulsion System (SEPS). Reference 1 investigates the early portion of this mission, a fast flyby of Comet Halley, including delivery of an atmospheric probe. Reference 2 investigates later parts of the mission, including a powered heliocentric cruise phase, an approach to rendezvous with Comet Tempel 2, and subsequent operation near the cometary nucleus. Information regarding the scientific objectives, mission design, and spacecraft design for this mission is presented in Refs. 1-3. Included in Ref. 2 are estimated orbit determination and guidance accuracies for the Tempel 2 rendezvous approach phase of the mission, including a number of sensitivity studies involving parameters such as data frequencies, data accuracies, ion drive thrust vector errors, comet ephemeris uncertainties, and time lags associated with data processing and command sequence generation. The rendezvous accuracies stated in Ref. 2 are substantially better than those stated in Ref. 4 for another ion drive comet rendezvous mission. In this paper, the approach to rendezvous with Tempel 2 is re-examined. Rendezvous accuracies significantly better than those presented in Ref. 2 are obtained. The parametric sensitivity results of Ref. 2 are updated. In addition, the sensitivity of the rendezvous accuracy to the limits placed on thrust vector changes for guidance purposes is investigated, along with the sensitivity to guidance mode (fixed vs variable time of arrival). Although this particular comet rendezvous mission is no longer under consideration by NASA for a new program start, due to funding limitations, the mission is still of interest, since it is representative of SEP comet rendezvous missions in general.

## Navigation Strategy

The navigation system for this mission consists of the Deep Space Network (DSN), which provides radiometric tracking data, elements of the flight system (imaging science subsystem, ion drive thrusters, and perhaps a radar altimeter or three-axis accelerometer), ground-based astronomical observatories, and ground-based computational facilities and software.[1,2] The Tempel 2 rendezvous approach phase begins 60 days before initial rendezvous, at which time the comet should be detectable with the narrow angle imaging system on board the spacecraft. Navigation tracking cycles[1] will be scheduled every five days throughout the approach phase. A single-station pass of two-way coherent doppler data, with ranging, will be needed each day, outside of the navigation tracking cycles. Ground-based comet observations will be

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